Lösungen HM1:

(1) z.z.: $\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$

I.A.: $n=1$: $-1 = -1$ ✓

I.S.: $n \rightsquigarrow n+1$: $\sum_{k=1}^{n+1} (-1)^k k^2 \overset{\text{I.V.}}{=} (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)(n+1)$

$= (-1)^{n+1} \left(\frac{-n(n+1)}{2} + 2 \cdot \frac{n(n+1)+(n+1)}{2} \right)$

$= (-1)^{n+1} \left(\frac{n(n+1)+2(n+1)}{2} \right) = (-1)^{n+1} \frac{(n+1)(n+2)}{2}$ //

(2) $b_k := \frac{1}{a_1 \cdot \ldots \cdot a_k}$, $0 \le b_k \le \frac{1}{2^k} \xrightarrow{k\to\infty} 0$

$\Rightarrow b_k \xrightarrow{k\to\infty} 0$; $b_0 := 1$

$\sum_{k=1}^{\infty} \frac{a_k - 1}{a_1 \cdot \ldots \cdot a_k} = \sum_{k=1}^{\infty} \left(\frac{1}{a_1 \cdot \ldots \cdot a_{k-1}} - \frac{1}{a_1 \cdot \ldots \cdot a_k} \right)$

$= \sum_{k=1}^{\infty} (b_{k-1} - b_k) \overset{\text{Teleskop}}{=} \lim_{n\to\infty} b_0 - b_{n+1} = 1$ //

(3) a) $\frac{\sinh x - 1}{\cosh x + 1} = \frac{\frac{e^x - e^{-x}}{2} - 1}{\frac{e^x + e^{-x}}{2} + 1} = \frac{\frac{1-e^{-2x}}{2} - e^{-x}}{\frac{1+e^{-2x}}{2} + e^{-x}} \xrightarrow{x\to\infty} \frac{\frac{1}{2}}{\frac{1}{2}} = 1$ //

b) $\frac{\sqrt{\cos(ax)} - \sqrt{\cos(bx)}}{x^2} = \frac{\cos(ax) - \cos(bx)}{x^2(\sqrt{\cos(ax)} + \sqrt{\cos(bx)})}$

$= \frac{1 - a^2x^2 - \left(1 - \frac{b^2x^2}{2}\right) + \sigma(x^2)}{x^2(\sqrt{\cos(ax)} + \sqrt{\cos(bx)})}$

$= \frac{b^2 - a^2}{2(\sqrt{\cos(ax)} + \sqrt{\cos(bx)})} + \sigma(1) \xrightarrow{x\to 0} \frac{b^2-a^2}{4}$ //

I (3) c) $\lim_{x\to 1} \frac{x^x - x}{1 - x + \ln x} \overset{\text{l'Hospital}}{=} \lim_{x\to 1} \frac{(\ln x + 1)x^x - 1}{-1 + \frac{1}{x}}$

$\overset{\text{l'Hospital}}{=} \lim_{x\to 1} \frac{((\ln x + 1)^2 + \frac{1}{x})x^x}{-\frac{1}{x^2}} = \frac{2}{-1} = -2$ //

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(4) a) $\forall n \ge 2$: $1 \le \sqrt[2n-3]{n-1} \le \sqrt[2n-3]{2n-3} \xrightarrow{n\to\infty} 1$

$\Rightarrow$ Konvergenzradius $r = 1$

Randpunkte: $|x| = 1$: $|(n-1)x^{2n-3}| = n-1 \underset{n\to\infty}{\not\to} 0 \Rightarrow$ Reihe diverg.

$\to$ Konvergenzmenge $(-1,1)$ //

b) $|x| < 1$: $\sum_{n=2}^{\infty} (n-1)x^{2n-3} = \frac{1}{2} \frac{d}{dx} \left(\int_0^x \sum_{n=2}^{\infty} (2n-2)\, t^{2n-3}\, dt \right)$

$= \frac{1}{2} \frac{d}{dx} \left(\sum_{n=2}^{\infty} \int_0^x (2n-2) \cdot t^{2n-3}\, dt \right)$

$= \frac{1}{2} \frac{d}{dx} \left(\sum_{n=2}^{\infty} x^{2n-2} \right) = \frac{1}{2} \frac{d}{dx} \left(x^2 \sum_{n=0}^{\infty} (x^2)^n \right)$

$= \frac{1}{2} \frac{d}{dx} \left(\frac{x^2}{1-x^2} \right) = \frac{1}{2} \frac{d}{dx} \left(-1 + \frac{1}{1-x^2} \right)$

$= \frac{1}{2} \cdot \frac{2x}{(1-x^2)^2} = \frac{x}{(1-x^2)^2}$ //

(5) $f(x) \xrightarrow{x\to\infty} \infty$, $f(x) \xrightarrow{x\to -\infty} -\infty$, f stetig

$\overset{\text{ZWS}}{\Rightarrow}$ $f(\mathbb{R}) = \mathbb{R}$; $f'(x) = 2e^{2x} + 3e^{-3x} > 0$, $x \in \mathbb{R}$

$\Rightarrow f$ injektiv. Insgesamt: $f: \mathbb{R} \to \mathbb{R}$ bijektiv,

also $\exists$ Umkehrfkt. $g: \mathbb{R} \to \mathbb{R}$; $f(0) = 0 \Rightarrow g(0) = 0$

$g'(y) = \frac{1}{f'(g(y))}$; $f'(x) = 2e^{2x} + 3e^{-3x}$; also

$f'(0) = 5$, also $g'(0) = \frac{1}{f'(0)} = \frac{1}{5}$

6. a) $f_n(x) = \sin\left(\frac{x}{n}\right) \xrightarrow{n\to\infty} \sin 0 = 0$ für alle $x \in \mathbb{R}$, aber für $x_n = n\frac{\pi}{2}$ gilt:

$|f_n(x_n)| = |\sin\frac{\pi}{2}| = 1 \underset{\not\to}{\xrightarrow{n\to\infty}} 0$.

Also (f_n) nicht gleichmäßig konvergent auf $\mathbb{R}$.

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7. $x^2 + x - 1 = A(x^2+1) + Bx^2 + Cx$; $x = 0 \rightsquigarrow A = -1$

$x = 1$; $x = -1$: $\rightsquigarrow \left.\begin{array}{l} 1 = -2 + B + C \\ -1 = -2 + B - C \end{array}\right] \rightsquigarrow$

$\rightsquigarrow \quad C = 3 - B \quad \rightsquigarrow \quad C = 1$

$\quad 2B = 4 \qquad\qquad B = 2$

also: $\int \frac{x^2+x-1}{x^3+x}\,dx = \int -\frac{1}{x} + \frac{2x+1}{x^2+1}\,dx$

$= \int -\frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

$\overset{x>0}{=} -\ln x + \ln(x^2+1) + \arctan x + C$

$= \ln\frac{x^2+1}{x} + \arctan x + C$

Lösungen HM2:

1. a) Sei $((x_n, y_n))$ Folge mit $(x_n, y_n) \xrightarrow{n\to\infty} (x,y) \in \mathbb{R}^2$

1. Fall $(x,y) \neq (0,0)$: $\exists n_0 \in \mathbb{N}$: $(x_n, y_n) \neq (0,0)$ $n \geq n_0$

$\Rightarrow \forall n \geq n_0$: $f(x_n, y_n) = \frac{x_n^3}{x_n^2+y_n^2} \xrightarrow{n\to\infty} \frac{x^3}{x^2+y^2} = f(x,y)$

2. Fall $(x,y) = (0,0)$: OBdA: $\forall n$ $(x_n, y_n) \neq (0,0)$

$\rightsquigarrow |f(x_n, y_n)| = \frac{|x_n|^3}{(x_n^2+y_n^2)} \leq |x_n| \xrightarrow{n\to\infty} 0 = f(0,0)$

$\Rightarrow f(x_n, y_n) \xrightarrow{n\to\infty} 0 = f(0,0)$

b) $e = (e_1, e_2)$: $\frac{\partial f}{\partial e}(0,0) = \lim_{t\to 0} \frac{f(te) - f(0)}{t}$

$= \lim_{t\to 0} \frac{\frac{t^3}{t^2} f(e)}{t} = f(e) = e_1^3$

c) $\operatorname{grad} f(0,0) = \left(\frac{\partial f}{\partial(1,0)}(0,0), \frac{\partial f}{\partial(0,1)}(0,0)\right)^T$

$\overset{b)}{=} (f(1,0), f(0,1))^T = (1,0)^T$; für $e = \frac{1}{\sqrt{2}}(1,1)$:

$e \cdot \operatorname{grad} f(0,0) = \frac{1}{\sqrt{2}} \neq \left(\frac{1}{\sqrt{2}}\right)^3 = f(e) \overset{b)}{=} \frac{\partial f}{\partial e}(0,0)$

$\Rightarrow f$ nicht diff.bar in $(0,0)$.

d) $\operatorname{grad} f(0,0) \overset{c)}{=} (1,0)^T$; $(x,y) \neq (0,0)$:

$\operatorname{grad} f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)^T$

$= \left(\frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2}, -\frac{2yx^3}{(x^2+y^2)^2}\right)^T$

$= \left(\frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}, -\frac{2yx^3}{(x^2+y^2)^2}\right)^T$

II **2.** T kompakt, f stetig $\Rightarrow$ Min und Max von $f|_T$ ex.

<u>Stationäre Punkte im Innern:</u>

$\operatorname{grad} f(x,y) = (y-4x, x-4y)^T \stackrel{!}{=} (0,0)^T$

$\Rightarrow (x,y) = (0,0)$

Rand: Nach Lagrange existiert $\lambda \in \mathbb{R}$ mit

$(f+\lambda g)'(x,y,\lambda) = 0$, wobei $g(x,y) = x^2+y^2-8$

also: $y - 4x + 2\lambda x = 0$ (1);

$x - 4y + 2\lambda y = 0$ (2)

$x^2 + y^2 - 8 = 0$ (3)

(1)+(2): $(x+y) - 4(x+y) + 2\lambda(x+y) = 0$

$\Rightarrow x+y = 0 \vee \lambda = \frac{3}{2}$

<u>1. Fall: $\lambda = \frac{3}{2}$</u>: (1) $\Rightarrow x = y$; (3) $\Rightarrow x = y = \pm 2$

<u>2. Fall: $x = -y$</u>: (3) $\Rightarrow (x,y) = \pm(2,-2)$

alle in Frage kommenden Punkte:

$f(0,0) = 0$; $f(\pm 2, \pm 2) = -12$; $f(\pm 2, \mp 2) = -20$

$\Rightarrow \min f|_T = -20$, $\max f|_T = 0$.

3. $(x,y,z) = r(\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$,

$d(x,y,z) = r^2\cos\theta \, d(r,\varphi,\theta)$,

$$\int_{B_\rho} \frac{1}{\sqrt{x^2+y^2+z^2}} \, d(x,y,z) = \int_\rho^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^2}{r} \cos\theta \, d(\theta,\varphi,r)$$

$$= \frac{\pi}{2} \int_\rho^R r \, dr = \frac{\pi}{4}(R^2-\rho^2) \xrightarrow{\rho \to 0^+} \frac{\pi}{4} R^2$$

4. <u>Bernoulli-Dgl.</u> $(-2)y^{-3}$

$\leadsto (y^{-2})' - 4x^3 - 4xy^{-2} = 0$. Subst. $u = y^{-2}$

(gesucht $y > 0$).

$u' = 4xu + 4x^3$. <u>lineare Dgl.</u>, also:

$u_h = Ce^{\int 4x\,dx} = Ce^{2x^2}$

<u>Ansatz</u>: $u_p(x) := C(x)e^{2x^2}$

$\leadsto u_p'(x) = C'(x)e^{2x^2} + C(x)4xe^{2x^2} \stackrel{!}{=} 4xC(x)e^{2x^2} + 4x^3$

$\leadsto C'(x) = 4x^3e^{-2x^2}$

$\leadsto C(x) = \int 4x^3e^{-2x^2}dx = -x^2e^{-2x^2} + \int 2xe^{-2x^2}dx$

$= -x^2e^{-2x^2} - \frac{1}{2}e^{-2x^2}$

allg. Lsg.: $u = -x^2 - \frac{1}{2} + Ce^{2x^2}$;

$u(0) = y(0)^{-2} \stackrel{!}{=} 1 \Rightarrow C = \frac{1}{2}$

$\stackrel{y>0}{\Rightarrow} y = \frac{1}{\sqrt{u}} = \frac{\sqrt{2}}{\sqrt{e^{2x^2}-1-2x^2}}$

5. a) <u>lineare Dgl.</u>: $y_h = Ce^{\int \tan x\,dx} = Ce^{-\ln(\cos x)}$

$= \frac{C}{\cos x}$; Ansatz: $y_p(x) := \frac{C(x)}{\cos(x)}$

$\Rightarrow y_p' = \frac{C'(x)}{\cos(x)} + C(x)\frac{\sin(x)}{(\cos(x))^2} \stackrel{!}{=} C(x)\frac{\sin x}{(\cos x)^2} - 2\sin x$

$\leadsto C'(x) \stackrel{!}{=} -2\sin x\cos x = -\sin(2x)$.

$\leadsto C(x) = \frac{\cos(2)}{2} \rightarrow$ allg. Lsg.: $y = \frac{C}{\cos x} + \frac{\cos x}{2} - \frac{\sin x \sin x}{2}$

$= \frac{C}{\cos x} + \frac{\cos(2x)}{2\cos x}$

b) <u>getrennte Veränderliche</u>: $\int e^{-y}dy = \int \sin x\,dx = C - \cos x$

$\Rightarrow e^{-y} = \cos x - C \Rightarrow y = -\ln(\cos x - C)$

Lösung auf $(-\frac{\pi}{2}, +\frac{\pi}{2})$, falls $C \le 0$.

II **6.** $\lambda^4 + 2\lambda^2 + 1 = (\lambda^2+1)^2 \stackrel{!}{=} 0$

$\Leftrightarrow (\lambda - i)^2(\lambda + i)^2 = 0$

$\rightsquigarrow u_h = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$.

Ansatz: $u_p = a\cos(2t) + b\sin(2t)$

$\rightsquigarrow 16u_p - 8u_p + u_p \stackrel{!}{=} \sin(2t)$

$\rightsquigarrow 9a\cos(2t) + 9b\sin(2t) \stackrel{!}{=} \sin(2t)$

$\rightarrow a = 0,\ b = 1/9$

allg. Lsg.: $u = c_1\cos t + c_2 \sin t + c_3 t\cos t + c_4 t \sin t + \frac{1}{9}\sin(2t)$.

$u(0) \stackrel{!}{=} 0 \Rightarrow c_1 = 0$; $u'(0) = 0 \Rightarrow c_2 + c_3 + \frac{2}{9} = 0\ (*)$;

$u''(0) = 0 \Rightarrow 2c_4 = 0 \Rightarrow c_4 = 0$;

$u'''(0) \stackrel{!}{=} 0 \Rightarrow -c_2 - c_3 \cdot 3 - \frac{8}{9} = 0 \stackrel{(*)}{\Rightarrow} c_3 = -\frac{1}{3}$

$c_2 = \frac{1}{9}$

$\Rightarrow u = \frac{1}{9}(\sin t - 3t\cos t + \sin(2t))$

7. a) komplexe Form: $\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$

$= \frac{1}{2\pi}\left(\int_{-\pi}^{0} \alpha x e^{-ikx}\,dx + \int_0^{\pi} \beta x e^{-ikx}\,dx\right)$

Fall $k=0$: $\hat{f}(0) = \frac{1}{2\pi}\left(\frac{\alpha x^2}{2}\Big|_{x=-\pi}^{0} + \frac{\beta x^2}{2}\Big|_{x=0}^{\pi}\right) = \frac{(\beta-\alpha)\pi}{4}$

Fall $k\neq 0$: $\int x e^{-ikx}\,dx = \frac{xe^{-ikx}}{-ik} - \int \frac{e^{-ikx}}{-ik}\,dx$

$= \left(\frac{ix}{k} + \frac{1}{k^2}\right)e^{-ikx}$, also:

$2\pi\hat{f}(k) = \alpha\left(\frac{ix}{k} + \frac{1}{k^2}\right)e^{-ikx}\Big|_{x=-\pi}^{0} + \beta\left(\frac{ix}{k} + \frac{1}{k^2}\right)e^{-ikx}\Big|_{x=0}^{\pi}$

$= \frac{\alpha}{k^2} - \alpha\left(-\frac{i\pi}{k} + \frac{1}{k^2}\right)e^{-ik\pi} + \beta\left(\frac{i\pi}{k} + \frac{1}{k^2}\right)e^{-ik\pi} - \frac{\beta}{k^2}$

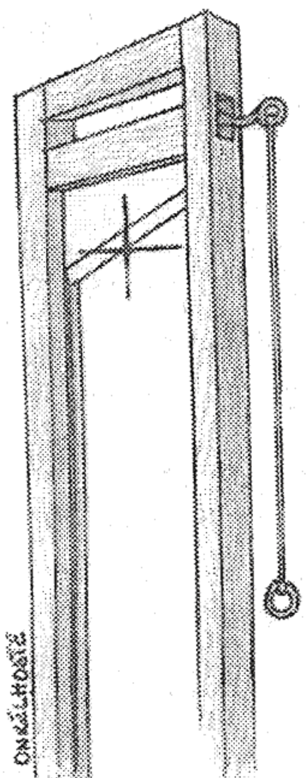
$e^{ik\pi} = e^{-ik\pi} = (-1)^k$

$\rightsquigarrow \hat{f}(0) = \frac{(\beta-\alpha)\pi}{2}$; $a_k = \hat{f}(k) + \hat{f}(-k) = \frac{(\alpha-\beta)(1-(-1)^k)}{\pi k^2}$;

II (7. a) forts.)

$b_k = -\frac{(-1)^k(\alpha+\beta)}{k} = i(\hat{f}(k) - \hat{f}(-k))$ $(k\neq 0)$

b) f stückweise glatt $\Rightarrow f$ wird durch Fourierreihe in allen Stetigkeitspunkten von f dargestellt. Das sind $x \in \mathbb{R}\setminus\{(2k+1)\pi : k \in \mathbb{Z}\}$ falls $\alpha \neq -\beta$ und $x \in \mathbb{R}$ falls $\alpha = -\beta$.



"7 TAGE - 7 KÖPFE"
Ich mochte die Show, obwohl sie ein vorhersehbares Ende hatte...